(5) *Summary of Results.*

(*a*) The following doublet separations were found for the first four members of the Balmer series $H_\alpha = 0{\cdot}154$ Å.U., $H_\beta = 0{\cdot}085$ Å.U., $H_\gamma = 0{\cdot}062$ Å.U. and $H_\delta = 0{\cdot}049$ Å.U.

When these separations were plotted against the squares of the corresponding wave-lengths they were shown to lie on a curve which pointed to the vanishing of the doublet separation at the short wave-length limit of the Balmer series.

(*b*) The wave-lengths H_α, H_β, H_γ and H_δ have been shown to consist of triplets when the electric discharge takes place in hydrogen containing a trace of nitrogen.

(*c*) Experiments made to decide whether the third members of the triplets had their origin in the radiations emitted by nitrogen atoms or in a modification produced by the nitrogen in the radiations emitted by the atoms of hydrogen were not decisive.

A Problem in the Theory of Heat Conduction.

By J. W. Nicholson, F.R.S.

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The first problem discussed in this paper is as follows:—" A long hollow cylinder is immersed in a medium of uniform temperature, and at a certain instant, is brought suddenly to a temperature above that of the medium, for example, by the passage through it of a stream of heated liquid. It is *maintained* at that temperature, and the temperature distribution in the outside medium at any instant, and the rate of leakage of heat from the cylinder, are required."

The main difficulty arises from the fact that the law of temperature on the surface of separation is arbitrary—we shall suppose, in the first instance, that it is constant. Solutions for the external temperature distribution are apparently known only when the surface has a given *initial* distribution, which is left to adjust itself without further direct supply of heat.

A solution for the distribution inside a cylinder whose surface is kept at zero temperature, and whose interior has any initial distribution, has been known for some time, and may be obtained by several methods. Perhaps the two most powerful methods of attack on problems of heat conduction are the synthetical method,* which replaces the actual distribution by sets of sources and sinks distributed over the boundaries and throughout the region to be investigated, and the method which makes use of Green's functions. Naturally, it is in the theory of Potential that these functions are especially valuable, and their extension to problems of heat conduction, apparently noticed first by Minnigerode in his 'Göttingen Dissertation' of 1862, and since developed in some detail, notably by Betti and Carslaw, is still by no means completely worked out. Carslaw has, by a method dependent on contour integration, determined their forms in many interesting cases.

We may define the Green's function, after Carslaw, as "the temperature at a point (x, y, z) at time t, due to an instantaneous point source generated at (x_0, y_0, z_0) at time τ, the bounding surface of the medium being maintained at zero temperature." With the help of the function so defined, we may show that

$$[v_{\mathrm{P}}]_t = \iiint [u]_{\tau=0} f(x, y, z)\, dx\, dy\, dz + k\int_0^t d\tau \left[\iint \phi(x, y, z, \tau)\, \frac{\partial u}{\partial n_2} d\mathrm{S}\right],$$

where $[v_{\mathrm{P}}]_t$ is the temperature at a point, P, or (x_0, y_0, z_0) at time t, due to an initial temperature distribution $f(x, y, z)$ at time $t = 0$, and the boundary, S, of the medium being kept at temperature $\phi(x, y, z, t)$ at any subsequent time.

The thermometric conductivity of the medium is k, the equation of heat propagation being

$$\frac{\partial u}{\partial t} = k\nabla^2 u$$

and u, whose differentiation in the second integral is along the inward normal, is the function

$$u = \{4\pi k(t-\tau)\}^{-3/2} e^{-\mathrm{R}^2/4k(t-\tau)},$$

where R is the distance between (x, y, z) and (x_0, y_0, z_0). The formula is not quite identical with that of Carslaw in his main paper, where an error of sign appears to be present.

A direct proof that this expression satisfies the differential equation, with the initial and boundary conditions, is not difficult.

With similar notation, the corresponding solution for the two-dimensional problem, in which everything is independent of the co-ordinate z, becomes

$$[v_{\mathrm{P}}]_t = \iint u_{\tau=0} f(x, y)\, dx\, dy + k\int_0^t d\tau \left[\iint \frac{\partial u}{\partial n_2} \phi(x, y, \tau)\, d\mathrm{S}\right],$$

the line integral being taken over the bounding arc, and the surface integral over any section of the medium perpendicular to z. The function u is now to be defined by

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* Hobson, 'Proc. Lond. Math. Soc.,' vol. 19.

$$u = \{4\pi k(t-\tau)\}^{-1} e^{-\mathrm{R}^2/4k(t-\tau)}$$

with $$\mathrm{R}^2 = (x-x_0)^2+(y-y_0)^2 = r^2+r'^2-2rr' \cos(\theta-\theta')$$

in terms of cylindrical co-ordinates whose axis is z, so that (r, r') denote normal distances from the axis, and (θ, θ') the orientations in the normal section, measured from some fixed initial line. The Green's function employed in this case is the temperature at a point (x, y) at time t due to a line source generated at the instant τ along $x = x_0$, $y = y_0$, in the medium, the bounding surface being kept at zero temperature. In the application of the theorem, there are some difficulties which have not received attention, but these do not occur in this special case.

The Cylindrical Source.

There is a simple expression for the temperature at any point at time t due to an instantaneous source of heat generated on the surface of a circular cylinder at $t = 0$, which has not received any specific notice. The corresponding formula for a point source in three dimensions, situated at the origin, is

$$u = \mathrm{Q}(4\pi kt)^{-3/2} e^{-\mathrm{R}^2/4kt}$$

at distance $\mathrm{R} = \sqrt{(x^2+y^2+z^2)}$, where Q is the quantity of heat instantaneously generated and then left to itself. For a line source along the line $x = x'$, $y = y'$, with a quantity Q per unit length,

$$u = \mathrm{Q}(4\pi kt)^{-1} e^{-[(x-x')^2+(y-y')^2]/4kt},$$

the source being created in each case at time $t = 0$. Consider now an infinitely long cylindrical source, with a quantity Q generated per unit length. The radius is a, and θ is an orientation in any normal section. The quantity of heat generated in the strip of unit length between two generators subtending $\delta\theta$ on the axis is

$$\mathrm{Q}\delta\theta/2\pi$$

and this can be treated as a line source. By summation, the subsequent temperature due to the cylindrical source is

$$u = (4\pi kt)^{-1} \int_0^{2\pi} \frac{\mathrm{Q}d\theta}{2\pi} e^{-(\rho^2+a^2-2\rho a\cos\theta)/4kt}$$

at distance ρ from the axis. This becomes

$$u = \frac{\mathrm{Q}}{4\pi kt} \mathrm{I}_0\left(\frac{\rho a}{2kt}\right) e^{-(\rho^2+a^2)/4kt},$$

where I_0 is the Bessel function defined by

$$\mathrm{I}_0(x) = \mathrm{J}_0(ix).$$

Cylindrical Surface Maintained at Temperature v_0.

We now pass to the problem of more special interest, in which a temperature, v_0, which is constant, is established at $t = 0$ over a cylindrical surface, $\rho = a$, and *maintained* at the value v_0.

The general theorem is not now applicable, for the space is not simply connected outside. The equation of conduction is modified at the surface, by the continual creation of heat. It is, moreover, different inside on account of the difference of k, the preceding theorem involving the same value of k throughout. Since the internal space is shut off by an arbitrarily prescribed surface temperature, the internal value of k cannot affect the external problem, and we might at first sight expect to find a theorem enabling us to write down the solution for the doubly-connected external space. But we shall find later that this is apparently impossible.

The general theorem will first be applied to the internal space, and the form of the result will suggest an appropriate type of external solution. This appears to be the first solution of its kind for the space outside a cylinder and by its nature we shall find it possible to determine the general solution corresponding to an arbitrarily maintained law of surface temperature.

The internal solution, composed of parts due to (1) the prescribed surface rigidity of distribution, and (2) the initial temperature, say v_0, from $\rho = 0$ to $\rho = a$, may now be written down. It should be $v = v_0$ permanently.

Denoting the parts of v corresponding to (1) and (2) by v_1, v_2 respectively, we have

$$v_1 = -k\int_0^t d\tau \int \frac{\partial u}{\partial n_2} v_0\, ds,$$

$$v_2 = \iint u_{\tau=0}\, v_0\, dx\, dy.$$

The outward normal to the surface is along a increasing, and therefore

$$\partial/\partial n_2 = -\partial/\partial a,$$

and with

$$u = \{4\pi k(t-\tau)\}^{-1} e^{-\mathrm{R}^2/4k(t-\tau)}$$

we have

$$v_1 = -kv_0\int_0^t d\tau\, \{4\pi k(t-\tau)\}^{-1}\int_0^{2\pi} d\theta\, .\, a\frac{\partial}{\partial a}\, .\, e^{-\mathrm{R}^2/4k(t-\tau)}.$$

But

$$\mathrm{R}^2 = \rho^2+a^2-2a\rho\cos\theta, \qquad \partial\mathrm{R}^2/\partial a = 2(a-\rho\cos\theta),$$

and

$$v_1 = \frac{av_0}{8\pi k}\int_0^t \frac{d\tau}{(t-\tau)^2}\int_0^{2\pi} d\theta\, (a-\rho)\cos\theta)\, e^{-\mathrm{R}^2/4k(t-\tau)},$$

or with a new variable β defined by $4k\beta(t-\tau) = 1$,

$$v_1 = \frac{av_0}{2\pi}\int_{\beta_0}^{\infty} d\beta \int_0^{2\pi} d\theta\,(a-\rho\cos\theta)\,e^{-\beta(\rho^2+a^2-2a\rho\cos\theta)}$$

where $$\beta_0 = 1/4kt.$$

This evidently becomes

$$v_1 = av_0\int_{\beta_0}^{\infty} d\beta\,\{a\mathrm{I}_0(2\rho a\beta)-\rho\mathrm{I}_1(2\rho a\beta)\}\,e^{-\beta(\rho^2+a^2)}.$$

This can be reduced by integration by parts, to

$$v_1 = \tfrac{1}{2}v_0\mathrm{I}_0\left(\frac{\rho a}{2kt}\right)e^{-(\rho^2+a^2)/4kt}-\frac{v_0}{2}(\rho^2-a^2)\int_{\beta_0}^{\infty} d\beta\,\mathrm{I}_0(2\rho a\beta)\,e^{-\beta(\rho^2+a^2)},$$

the first term being its limiting value on the surface $\rho = a$.

Again the effect of the initial volume condition,—temperature v_0 throughout the cylinder,—is

$$v_2 = v_0\iint u_{\tau=0}\,dx\,dy,$$

the integration being over an internal section. Thus

$$v_2 = \frac{v_0}{2kt}\int_0^a \sigma d\sigma \int_0^{2\pi} d\theta\,.\,e^{-(\rho^2+\sigma^3-2\rho\sigma\cos\theta)/4kt},$$

$$= \frac{v_0}{2kt}\int_0^a \sigma\,d\sigma\,\mathrm{I}_0\left(\frac{\rho\sigma}{2kt}\right)e^{-(\rho^2+\sigma^2)/4kt}.$$

The complete internal solution is $v = v_1+v_2$, or

$$v = \frac{v_0}{2kt}\int_0^a \sigma d\sigma\,\mathrm{I}_0\left(\frac{\rho\sigma}{2kt}\right)e^{-(\rho^2+\sigma^2)/4kt}$$
$$+av_0\int_{(4kt)^{-1}}^{\infty} d\beta\,\{a\mathrm{I}_0(2\rho a\beta)-\rho\mathrm{I}_1(2\rho a\beta)\}\,e^{-\beta(\rho^2+a^2)}.$$

The two integrals concerned in this formula are of very different types, and have not hitherto been related in the literature concerned with the Bessel functions. Nevertheless, there is clearly a remarkable relation between them, for the solution must be $v = v_0$.

We shall apply the Bessel-Fourier double integral to establish this connection, and to suggest the external solution. In the meantime, we notice that this theory of heat conduction gives a proof of a historical theorem of Weber. For the effect of a distribution of temperature $v = v_0$ at all points of space initially must be a temperature v_0 for all time. Accordingly, extending the integral in v_2 to infinity, we must have

$$2kt = \int_0^{\infty} \sigma d\sigma\,\mathrm{I}_0\left(\frac{\rho\sigma}{2kt}\right)e^{-(\rho^2+\sigma^2)/4kt},$$

or if $$p^2 = 1/4kt, \qquad m = \rho/2kt,$$

$$\int_0^\infty \sigma d\sigma \, \mathrm{I}_0(m\sigma)\, e^{-\rho^2\sigma^2} = \frac{1}{2p^2}\, e^{m^2/4p^2},$$

which is Weber's formula.

Application of the Bessel-Fourier Integral.

Let us consider the formula

$$u = \int_0^\infty z \mathrm{J}_0(\rho z)\, dz\, e^{-z^2kt} \int_0^\infty r \mathrm{J}_0(rz) f(r)\, dr$$

which satisfies the equation of heat conduction

$$\partial u/\partial t = k\nabla^2 u$$

by direct substitution, and which, when $t = 0$, gives $u = f(\rho)$ by the Bessel-Fourier theorem. Clearly u is the temperature at any point of space at time t, due to the initial distribution, $f(\rho)$ depending only on ρ, established at $t = 0$ and left to itself.

Let us choose the initial distribution as a uniform temperature, v_0, from $\rho = 0$ to $\rho = a$, and a uniform zero temperature from $\rho = a$ to $\rho = \infty$. Then the subsequent temperature at any point is

$$u = \int_0^\infty z\, dz \mathrm{J}_0(\rho z)\, e^{-z^2kt} \int_0^a r\mathrm{J}_0(rz)\, v_0\, dr.$$

Since $f(r) = v_0$ from $r = 0$ to a, and zero beyond. But

$$\int_0^a r\mathrm{J}_0(rz)\, dr = \frac{a}{z}\mathrm{J}_1(az),$$

and therefore

$$u = av_0 \int_0^\infty dz \mathrm{J}_0(\rho z)\, \mathrm{J}_1(az)\, e^{-z^2kt}.$$

Since this must be identical with the previous v_2, we have the remarkable formula

$$\int_0^a \sigma d\sigma\, \mathrm{I}_0\left(\frac{\rho\sigma}{2kt}\right) e^{-(\rho^2+\sigma^2)/4kt} = 2akt \int_0^\infty dz\, \mathrm{J}_0(\rho z)\, \mathrm{J}_1(az)\, e^{-z^2kt}.$$

This can also be proved analytically as follows:—

If $f(\rho)$ is the integral on the left-hand side, it is equal, by the Bessel-Fourier theorem, to

$$f(\rho) = \int_0^\infty z\mathrm{J}_0(\rho z)\, dz \int_0^\infty r\mathrm{J}_0(rz)\, dr \int_0^a \sigma d\sigma\, \mathrm{I}_0\left(\frac{r\sigma}{2kt}\right) e^{-r^2/4kt}$$

on interchanging the last two integrations, as their nature allows,

$$f(\rho) = \int_0^\infty z\mathrm{J}_0(\rho z)\, dz \int_0^a \sigma d\sigma\, e^{-\sigma^2/4kt} \int_0^\infty r\mathrm{J}_0(rz)\, \mathrm{I}_0\left(\frac{r\sigma}{2kt}\right) e^{-r^2/4kt}.$$

The last integral is one of Weber's, having the value

$$2kt\,J_0(\sigma z)\,e^{-z^2kt+\sigma^2/4kt}$$

and thus

$$f(\rho) = 2kt\int_0^\infty z\,dz\,J_0(\rho z)\,e^{-z^2kt}\int_0^a \sigma\,d\sigma\,J_0(z\sigma)$$

$$= 2akt\int_0^\infty dz\,J_0(\rho z)\,J_1(az)\,e^{-z^2kt}$$

which demonstrates the identity.

Accordingly, in a preceding section,

$$v_2 = av_0\int_0^\infty dz\,J_0(\rho z)\,J_1(az)\,e^{-z^2kt}.$$

We next examine v_1: since by Weber's formula

$$I_0\left(\frac{\rho a}{2kt}\right)e^{-(\rho^2+a^2)/4kt} = 2kt\int_0^\infty z\,dz\,J_0(\rho z)\,J_0(az)\,e^{-z^2kt},$$

we may note, in passing, an elegant form for the effect of a cylindrical source. For an instantaneous generation of heat on a cylindrical surface $\rho = a$ at $t = 0$, of amount Q per unit length, the subsequent temperature at all internal or external points is

$$v = \frac{Q}{2\pi}\int_0^\infty z\,dz\,J_0(\rho z)\,J_0(az)\,e^{-z^2kt}.$$

Again

$$I_1\left(\frac{\rho a}{2kt}\right)e^{-(\rho^2+a^2)/4kt} = 2kt\int_0^\infty z\,dz\,J_1(\rho z)\,J_1(az)\,e^{-z^2ke},$$

and therefore

$$(aI_0-\rho I_1)\,e^{-(\rho^2+a^2)/4kt} = 2kt\int_0^\infty z\,dz\,\{aJ_0(\rho z)\,J_0(az)-\rho\,J_1(\rho z)\,J_1(az)\}\,e^{-z^2kt}.$$

But

$$\frac{\partial}{\partial z}\,.\,zJ_1(az) = azJ_0(az),$$

and therefore integrating by parts, the integrated terms vanishing

$$\int_0^\infty dz\,J_0(\rho z)\,J_0(az)\,e^{-z^2kt} = 2kt\int_0^\infty z^2\,dz\,J_0(\rho z)\,J_1(az)\,e^{-z^2kt}$$

$$+\rho\int_0^\infty dz\,J_1(\rho z)\,J_1(az)\,e^{-z^2kt},$$

so that, for all positive values of t,

$$(aI_0-\rho I_1)\,e^{-(\rho^2+a^2)/4kt} = 4k^2t^2\int_0^\infty z^2\,dz\,J_0(\rho z)\,J_1(az)\,e^{-z^2kt}$$

$$= \frac{1}{4\beta^2}\int_0^\infty z^2\,dz\,J_0(\rho z)\,J_1(az)\,e^{-z^2/4\beta}$$

with $4kt\beta = 1$.

Thus v_1, in preceding section, becomes

$$v_1 = av_0 \int_0^\infty z^2 dz \mathrm{J}_0(\rho z)\, \mathrm{J}_1(az) \int_{\beta_0}^\infty \frac{d\beta}{4\beta^2} e^{-z^2/4\beta} \qquad \left(\beta_0 = \frac{1}{2kt}\right)$$

$$= av_0 \int_0^\infty dz \mathrm{J}_0(\rho z)\, \mathrm{J}_1(az) \{1 - e^{-z^2kt}\},$$

and finally, $v_1 + v_2$ is given by the simple form

$$v_1 + v_2 = av_0 \int_0^\infty dz \mathrm{J}_0(\rho z)\, \mathrm{J}_1(az)$$

whose value is v_0 when ρ is less than a. We do not dwell upon some of the interesting formulæ of pure mathematics which can be derived from the above analysis.

The External Space.

Integrals of the types

$$\int_0^\infty dz f(z) \{\mathrm{J}_0(\rho z), \qquad \mathrm{Y}_0(\rho z)\} e^{-z^2kt}$$

are now clearly most suitable for this space, Y_0 being the second function defined by

$$\mathrm{Y}_0(x) = \left\{\frac{\partial \mathrm{J}_n(x)}{\partial n} - (-)^n \frac{\partial \mathrm{J}_{-n}(x)}{\partial n}\right\}_{n=0}$$

after Hankel, the function J_0 alone being clearly insufficient. Consider then the formula

$$v = v_0 - v_0 \int_0^\infty dz f(z) \{\mathrm{J}_0(\rho z)\, \mathrm{Y}_0(az) - \mathrm{Y}_0(\rho z)\, \mathrm{J}_0(az)\} e^{-z^2kt}.$$

This satisfies the differential equation, $f(z)$ not involving ρ or t. Certain restrictions on $f(z)$ may, of course, be necessary to secure convergency. The expression takes the value v_0 when $\rho = a$, for all time, so that the condition of *maintained* constant surface temperature is satisfied. It also takes the value v_0 at all points after an infinite time, as is necessary. It is therefore the solution we seek if it vanishes everywhere outside the surface $\rho = a$ at time $t = 0$, or, analytically, if

$$\int_0^\infty f(z)\, dz \{\mathrm{J}_0(\rho z)\, \mathrm{Y}_0(az) - \mathrm{Y}_0(\rho z)\, \mathrm{J}_0(az)\} = 1$$

for all values of ρ greater than a. This serves to determine $f(z)$.

The generalisation of the Bessel-Fourier theorem given by Orr* is peculiarly suitable for a problem of this type. Apparently it has never been applied as yet to any physical problem, although a procedure which is somewhat parallel, with series instead of integrals, can be traced in some of

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* 'Proc. R. Irish A.,' Sect. A, vol. 27, 1907–9.

Carslaw's papers. As enunciated by Orr, the theorem states that for values of ρ between a and b, the value of the integral

$$\int_0^\infty \lambda\, d\lambda \int_a^b r\phi(r)\, dr\, u_n(r, a)\, u_n(\rho, a)/\mathrm{K}_n(\iota\lambda a)\,\mathrm{K}_n(-\iota\lambda a)$$

where $$u_n(r, a) \equiv \mathrm{K}_n(\iota\lambda r)\,\mathrm{K}_n(-\iota\lambda a) - \mathrm{K}_n(-\iota\lambda r)\,\mathrm{K}_n(\iota\lambda a)$$

is $-\pi^2\phi(\rho)$ at ordinary points, and

$$-\tfrac{1}{2}\pi^2\{\phi(\rho-\epsilon)+\phi(\rho+\epsilon)\}_{\epsilon=0}$$

at a point ρ where ϕ may have a possible discontinuity, the mean on the two sides of the discontinuity being then required. There are no discontinuities in the space external to the cylinder with which we deal, so that the value will be $-\pi^2\phi(\rho)$ at the point ρ. The same integral is equal to zero when $\rho = a$, and to $-\frac{1}{2}\pi^2\phi(\rho-\epsilon)$ when $\rho = b$.

The function K_n in the above statement is the function which, in wave theory, is appropriate to a disturbance diverging from the origin, and is defined by

$$\frac{2\sin n\pi}{\pi}\,\mathrm{K}_n(\iota x) = \iota^n \mathrm{J}_{-n}(\iota x) - \iota^{-n}\mathrm{J}_n(\iota x).$$

When n becomes an integer, it is easy to evaluate K_n in terms of J and Y functions. We find that Orr's theorem takes the form, where ϕ is an arbitrary function,

$$\int_0^\infty \lambda\, d\lambda \int_a^b \frac{\{\mathrm{J}_n(\lambda\rho)\,\mathrm{Y}_n(\lambda a) - \mathrm{Y}_n(\lambda\rho)\,\mathrm{J}_n(\lambda a)\}\;\{\mathrm{J}_n(\lambda r)\,\mathrm{Y}_n(\lambda a) - \mathrm{Y}_n(\lambda r)\,\mathrm{J}_n(\lambda a)\}}{\mathrm{Y}_n^2(\lambda a) + \pi^2\mathrm{J}_n^2(\lambda a)}\, r\phi(r)\, dr,$$

$$\begin{aligned} &= \tfrac{1}{2}\{\phi(\rho-\epsilon)+\phi(\rho+\epsilon)\} && \text{if } b > \rho > a \\ &= 0 && \text{if } \rho = a \\ &= \tfrac{1}{2}\phi(\rho-\epsilon) && \text{if } \rho = b. \end{aligned}$$

In our problem, b will ultimately be infinite. We require $\phi(\rho) = 1$ between $\rho = a$ and $\rho = b$.

Putting $n = 0$, and identifying the above double integral with the statement

$$\int_0^\infty dz\, f(z)\,\{\mathrm{J}_0(\rho z)\,\mathrm{Y}_0(az) - \mathrm{Y}_0(\rho z)\,\mathrm{J}_0(az)\} = 1$$

we at once obtain

$$v = v_0 - v_0 \int_0^\infty \frac{\lambda\, d\lambda\, \{\mathrm{J}_0(\lambda\rho)\,\mathrm{Y}_0(\lambda a) - \mathrm{Y}_0(\lambda\rho)\,\mathrm{J}_0(\lambda a)\}}{\mathrm{Y}_0^2(\lambda a) + \pi^2\mathrm{J}_0^2(\lambda a)}\,\mathrm{F}(\lambda)\, e^{-\lambda^2 kt}$$

where $$\mathrm{F}(\lambda) = \int_a^b r\, dr\,\{\mathrm{J}_0(\lambda r)\,\mathrm{Y}_0(\lambda a) - \mathrm{Y}_0(\lambda r)\,\mathrm{J}_0(\lambda a)\}.$$

This is the general solution for the region, initially at zero, between two cylinders of radii (a, b), maintained since $t = 0$ at temperatures $(v_0, \frac{1}{2}v_0)$ respectively.

Removing the outer cylinder to infinity, or writing $b = \infty$, we obtain the solution of the original problem proposed. Thus, the process being justified by the convergent integral finally obtained, we write

$$F(\lambda) = \left(\int_0^\infty - \int_0^a\right) r\,dr\,\{J_0(\lambda r)\,Y_0(\lambda a) - Y_0(\lambda r)\,J_0(\lambda a)\}.$$

But

$$\int_0^a \lambda r J_0(\lambda r)\,dr = \Big[r J_1(\lambda r)\Big]_0^a = a J_1(\lambda a),$$

$$\int_0^a \lambda r Y_0(\lambda r)\,dr = \Big[r Y_1(\lambda r)\Big]_0^a = a Y_1(\lambda a) + \frac{2}{\lambda}$$

where the expression

$$Y_0(x) = 2J_0(x)\left\{\log\frac{x}{2} + \gamma\right\} - \frac{2}{x} - \frac{x}{2} - \ldots$$

with x small, has been used. Moreover, the integrals from zero to infinity are well known to converge to the values

$$\int_0^\infty \lambda r J_0(\lambda r)\,dr = 0, \qquad \int_0^\infty \lambda r\,Y_0(\lambda r)\,dr = \frac{2}{\lambda}$$

when λ is not itself zero.

Thus $$\lambda\,F(\lambda) = a J_0(\lambda a)\,Y_1(\lambda a) - a\,Y_0(\lambda a)\,J_1(\lambda a) = -\frac{2}{\lambda}$$

using a well known property of the functions.

More generally, with an upper limit, b, for the integrals,

$$\lambda\,F(\lambda) = -\frac{2}{\lambda} - b\{J_0(\lambda a)\,Y_1(\lambda b) - Y_0(\lambda a)\,J_1(\lambda b)\}.$$

Now, if the first value $(-2/\lambda)$ of $F(\lambda)$ is substituted in the integral for u, this integral is still convergent. The reason for this is that the exponential factor secures convergence at the upper limit, however small it may be, provided that it is not actually zero. The denominator $Y_0^2 + \pi^2 J_0^2$ can never be zero, and the numerator never large, since the Bessel functions are oscillatory. Moreover, near the lower limit, on expanding the Bessel functions, the integrand behaves like

$$-\frac{\log a - \log\rho}{\lambda(\log\lambda)^2}$$

and the integral of $\lambda^{-1}(\log\lambda)^{-2}$ is proportional to $(\log\lambda)^{-1}$, which *vanishes* with λ. Accordingly, the final integral is finite, and we are justified in removing the cylindrical boundary of radius b to infinity—this justification was not immediately obvious.

For the temperature outside the cylinder of radius a, maintained at v_0 from $t = 0$ onwards, we finally deduce the expression

$$v = v_0 + 2v_0 \int_0^\infty \frac{d\lambda}{\lambda} \cdot \frac{J_0(\lambda\rho) Y_0(\lambda a) - Y_0(\lambda\rho) J_0(\lambda a)}{Y_0^2(\lambda a) + \pi^2 J_0^2(\lambda a)} e^{-\lambda^2 kt}.$$

When $t = 0$, convergence is secured by the Bessel functions themselves at all points except $\rho = a$, where the integral becomes zero through a discontinuity. The generality of this mode of solution of problems of *maintained* temperature will be evident later. At the same time, the form of this solution is perhaps a sufficient indication that it will not in general be possible to find an analytical theorem for the external space, of a similar character to that quoted in connection with the Green's function for the internal space.

The Leakage of Heat from the Cylinder.

Let c be the specific heat and D the density of the external medium. Then the quantity of heat leaking by conduction from a length, l, of the cylinder in unit time is given by

$$h = -2\pi a c \mathrm{D} \left(\frac{\partial v}{\partial \rho}\right)_{\rho = a}.$$

We may find this differential coefficient from the previous expression for v if the result is a convergent integral. Conditions of continuity, which are necessary, are satisfied at all external points by the nature of the integral. Thus,

$$\left(\frac{\partial v}{\partial \rho}\right)_{\rho = a} = -2v_0 \int_0^\infty \frac{d\lambda \{J_1(\lambda a) Y_0(\lambda a) - Y_1(\lambda a) J_0(\lambda a)\}}{Y_0^2(\lambda a) + \pi^2 J_0^2(\lambda a)} e^{-\lambda^2 kt},$$

or since
$$J_1(\lambda a) Y_0(\lambda a) - Y_1(\lambda a) J_0(\lambda a) = -2/\lambda a$$
we find that the rate of loss of heat from length l is

$$h = 8\pi k c \mathrm{D} v_0 l \int_0^\infty \frac{d\lambda}{\lambda} \cdot \frac{e^{-\lambda^3 kt}}{Y_0^2(\lambda a) + \pi^2 J_0^2(\lambda a)}.$$

When $t = 0$, this is instantaneously infinite, as we should expect, on physical grounds, for $Y_0^2 + \pi^2 J_0^2$ behaves near the upper limit like $2\pi/\lambda a$—by the known asymptotic expansions of the functions concerned—but, for all subsequent time, h is finite, tending, as it should, to zero after an infinite time.

The total heat lost from a length, l, at a time, t, after the start is

$$\begin{aligned} \mathrm{H} &= \int_0^t h\,dt \\ &= 8\pi c \mathrm{D} v_0 l \int_0^\infty \frac{d\lambda}{\lambda^3} (1 - e^{-\lambda^2 kt}) / \{Y_0^2(\lambda a) + \pi^2 J_0^2(\lambda a)\} \\ &= 8\pi c \mathrm{D} v_0 l a^2 \int_0^\infty \frac{d\omega}{\omega^3} (1 - e^{-\omega^2 \beta}) / \{Y_0^2(\omega) + \pi^2 J_0^2(\omega)\} \end{aligned}$$

where $\beta = kt/a^2$.

In this form, the integral is a function only of β, and can be tabulated by quadratures, and the use of the standard tables, now existing, of $J_0(\omega)$, $Y_0(\omega)$. For practical purposes, it would be more convenient to possess an approximate formula for the heat lost between two epochs t_1, t_2, not very close to the starting point. This quantity of heat is given by

$$H' = 8\pi c D v_0 l a^2 \int_0^\infty \frac{d\omega}{\omega^3}\left(e^{-\omega^2\beta_1} - e^{-\omega^2\beta_2}\right) / \left\{Y_0^2(\omega) + \pi^2 J_0^2(\omega)\right\}.$$

Now if t be not too small, the magnitudes

$$(\beta_1, \beta_2) \equiv k/a^2\,(t_1, t_2)$$

will ordinarily be large, so that the exponentials secure that the important parts of the integral shall come from the range near $\omega = 0$. We may then write

$$J_0(\omega) = 1, \qquad Y_0(\omega) = 2\,(\log \omega/2 + \gamma)$$

where γ is Euler's constant. Thus approximately

$$H' = 8\pi c D v_0 l a^2 \int_0^\infty \frac{d\omega}{\omega^3}\left(e^{-\omega^2\beta_1} - e^{-\omega^2\beta_2}\right) \Big/ \left\{\pi^2 + 4\left(\log\frac{\omega}{2} + \gamma\right)^2\right\}$$

which can be calculated rapidly by quadratures. We do not pursue this problem beyond the point of obtaining an exact solution.

Cylindrical Surface maintained at any arbitrary Law of Temperature. Conduction into the External Space.

The mode of procedure towards a solution of this more general problem is now fairly clear. Suppose that an indefinitely long cylindrical surface is maintained from $t = 0$,—when everything was at zero temperature—at a temperature $F(t)$. It is required to find the temperature at an external point.

Let it be possible to find an integral expression of $F(t)$ in the form—α and β being constants—

$$F(t) = \int_\alpha^\beta f(\lambda)\, e^{-\lambda^2 kt}$$

where k is the thermometric conductivity of the external medium. The majority of ordinary functions admit this possibility. Then the function

$$v_1 = \int_\alpha^\beta f(\lambda) \frac{J_0(\lambda\rho)}{J_0(\lambda a)}\, e^{-\lambda^2 kt}\, d\lambda$$

is a solution of the equation of heat conduction, which takes the value $F(t)$ when $\rho = a$. It will constantly preserve the appropriate arbitrary temperature on the surface, but it presupposes an initial distribution of temperature

$$(v_1)_{t=0} = \int_\alpha^\beta f(\lambda) \frac{J_0(\lambda\rho)}{J_0(\lambda a)}\, d\lambda$$

at all external points $\rho > a$. This difficulty may be removed by the addition of another solution of the equation of heat conduction, of the form

$$v_2 = \int_0^\infty g(\lambda)\,\{\mathrm{J}_0(\lambda\rho)\,\mathrm{Y}_0(\lambda a) - \mathrm{Y}_0(\lambda\rho)\,\mathrm{J}_0(\lambda a)\}e^{-\lambda^2 kt}\,d\lambda$$

where $g(\lambda)$ is to be determined. This expression necessarily vanishes when $\rho = a$, and $g(\lambda)$ must be so chosen that at all other points, initially, the preceding distribution is cancelled. We thus choose $g(z)$ to satisfy the equation

$$\int_0^\infty dz g(z)\,\{\mathrm{J}_0(z\rho)\,\mathrm{Y}_0(za) - \mathrm{Y}_0(z\rho)\,\mathrm{J}_0(za)\} = -\int_a^\beta f(x)\,\frac{\mathrm{J}_0(\rho x)}{\mathrm{J}_0(ax)}\,dx$$

for all values of ρ greater than a.

But by Orr's formula, when ρ lies between a and infinity,

$$\mathrm{J}_0(x\rho) = \int_0^\infty z\,dz\,.\,\frac{\mathrm{J}_0(z\rho)\,\mathrm{Y}_0(za) - \mathrm{Y}_0(z\rho)\,\mathrm{J}_0(za)}{\mathrm{Y}_0^2(za) + \pi^2\mathrm{J}_0(za)}$$
$$\int_a^\infty r\,dr\,\mathrm{J}_0(rx)\,\{\mathrm{J}_0(rz)\,\mathrm{Y}_0(az) - \mathrm{Y}_0(rz)\,\mathrm{J}_0(az)\}.$$

Now, by a familiar procedure, we can prove directly from the differential equations for the functions that

$$\int_a^\infty r\mathrm{J}_0(xr)\,\mathrm{J}_0(zr)\,dr = \frac{a}{z^2-x^2}\,\{x\mathrm{J}_1(ax)\,\mathrm{J}_0(az) - z\,\mathrm{J}_1(az)\,\mathrm{J}_0(ax)\},$$

$$\int_a^\infty r\mathrm{J}_0(xr)\,\mathrm{Y}_0(zr)\,dr = \frac{a}{z^2-x^2}\,\{x\mathrm{J}_1(ax)\,\mathrm{Y}_0(az) - z\,\mathrm{Y}_1(az)\,\mathrm{J}_0(ax)\},$$

and accordingly, on reduction,

$$\int_a^\infty r\,dr\,\mathrm{J}_0(rx)\,\{\mathrm{J}_0(rz)\,\mathrm{Y}_0(az) - \mathrm{Y}_0(rz)\,\mathrm{J}_0(az)\}$$
$$= \frac{az}{z^2-x^2}\,\mathrm{J}_0(ax)\,\{\mathrm{Y}_1(az)\,\mathrm{J}_0(az) - \mathrm{J}_1(az)\,\mathrm{Y}_0(az)\}$$
$$= \frac{2\,\mathrm{J}_0(ax)}{x^2-z^2},$$

which is a comparatively simple expression. Substituting this in the double integrals we find, when $\rho > a$,

$$\frac{\mathrm{J}_0(x\rho)}{\mathrm{J}_0(xa)} = -2\int_0^\infty \frac{z\,dz}{x^2-z^2}\,.\,\frac{\mathrm{J}_0(z\rho)\,\mathrm{Y}_0(za) - \mathrm{Y}_0(z\rho)\,\mathrm{J}_0(za)}{\mathrm{Y}_0^2(za) + \pi^2\,\mathrm{J}_0^2(za)}$$

a very remarkable formula of a quite new type. The "principal value" of the integral with regard to the point $x = z$ is understood to be taken.

It is fairly clear that more detailed analysis is not in fact necessary or suitable at this point, for it would encumber the main argument, and we are only attempting to sketch the general method which is effective for such problems. We deduce at once that $g(z)$ must be defined by

$$\int_0^\infty dz\, g(z)\{\mathrm{J}_0(z\rho)\,\mathrm{Y}_0(za)-\mathrm{Y}_0(z\rho)\,\mathrm{J}_0(za)\}$$

$$= -\int_\alpha^\beta f(x)\,\frac{\mathrm{J}_0(x\rho)}{\mathrm{J}_0(xa)}\,dx$$

$$= 2\int_0^\infty \frac{z\,dz\{\mathrm{J}_0(z\rho)\,\mathrm{Y}_0(za)-\mathrm{Y}_0(z\rho)\,\mathrm{J}_0(za)\}}{\mathrm{Y}_0{}^2(za)+\pi^2\mathrm{J}_0{}^2(za)}\int_\alpha^\beta \frac{f(x)\,dx}{x^2-z^2}$$

whence we may select

$$g(z) = \frac{2z}{\mathrm{Y}_0{}^2(za)+\pi^2\mathrm{J}_0(za)}\int_\alpha^\beta \frac{f(x)\,dx}{x^2-z^2}$$

and then

$$v_2 = \int_0^\infty g(\lambda)\,d\lambda\,\{\mathrm{J}_0(\lambda\rho)\,\mathrm{Y}_0(\lambda a)-\mathrm{Y}_0(\lambda\rho)\,\mathrm{J}_0(\lambda a)\}\;e^{-\lambda^2kt}.$$

The whole solution of the problem of heat conduction may then be expressed in the following terms:—

If the whole of space is at zero temperature, and a distribution of temperature $v_0 = \mathrm{F}(t)$ is established at $t = 0$ suddenly, and afterwards maintained, over a cylindrical surface $\rho = a$, then the temperature at all external points at time t is given by

$$v = \int_\alpha^\beta d\lambda\, f(\lambda)\,\frac{\mathrm{J}_0(\lambda\rho)}{\mathrm{J}_0(\lambda a)}\,e^{-\lambda^2kt}$$

$$+\int_0^\infty \frac{2\lambda\,d\lambda\,\{\mathrm{J}_0(\lambda\rho)\,\mathrm{Y}_0(\lambda a)-\mathrm{Y}_0(\lambda\rho)\,\mathrm{J}_0(\lambda a)}{\mathrm{Y}_0{}^2(\lambda a)+\pi^2\mathrm{J}_0{}^2(\lambda a)}\,e^{-\lambda^2kt}\int_\alpha^\beta \frac{f(x)\,dx}{x^2-\lambda^2}$$

where $f(x)$ and the limits α, β are defined by

$$\mathrm{F}(t) = \int_\alpha^\beta f(x)\,e^{-x^2kt}$$

—or perhaps a group of integrals of this exponential type.

As a single illustration, let

$$\mathrm{F}(t) = \frac{t}{c^2+t^2}$$

where c is constant. Then since

$$\frac{t}{c^2+t^2} = \int_0^\infty e^{-\mu t}\cos\mu c\,d\mu$$

$$= 2k\int_0^\infty x\,dx\cos kcx^2\,e^{-x^2kt} \qquad (\mu = cx^2),$$

we have in this case,

$$\alpha = 0, \qquad \beta = \infty, \qquad f(x) = 2kx\cos kcx^2$$

and the solution which maintains the temperature

$$F(t) = \frac{t}{c^2+t^2}$$

for all time over $\rho = a$, is

$$v = 2k\int_0^\infty \lambda\, d\lambda \cos kc\lambda^2 \frac{J_0(\lambda\rho)}{J_0(\lambda a)} e^{-\lambda^2 kt}$$

$$+ 4k\int_0^\infty \frac{\lambda\, d\lambda \{J_0(\lambda\rho)\, Y_0(\lambda a) - Y_0(\lambda a)\, J_0(\lambda\rho)\}}{Y_0{}^2(\lambda a) + \pi^2 J^2(\lambda a)}\, e^{-\lambda^2 kt} \int_0^\infty \frac{x \cos kcx^2}{x^2-\lambda^2}\, dx$$

and these integrals can be much reduced.

Further illustrations, and further reduction of such a solution into other forms readily possible, do not seem necessary. Our object has been served by the indication of the general solution of problems involving a prescribed temperature variation over a cylinder.

This general solution can be made to apply to the internal space by using Orr's theorem between the limits zero and a. Any initial distribution which is not zero may be treated for both spaces by superposition.

A Study of Catalytic Actions at Solid Surfaces. Part VII.—*The Influence of Pressure on the Rate of Hydrogenation of Liquids in Presence of Nickel.*

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In Parts I, II and V* of this series we have indicated the reasons which lead us to believe that during the catalytic hydrogenation of liquids the function of the metallic catalyst is to combine with both agents—the unsaturated organic compound and hydrogen—and produce an unstable intermediate complex. The experimental evidence has furthermore impressed upon us the conviction that the determining factor in hydrogenation is the degree of affinity displayed between nickel and the unsaturated compound; whilst we have had ample opportunity to observe the selective nature of the process as exemplified by the widely-varying rates of absorption of hydrogen characteristic of various definite types of organic compounds, we have not

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